Periodic Thresholds and Rotations of Relations

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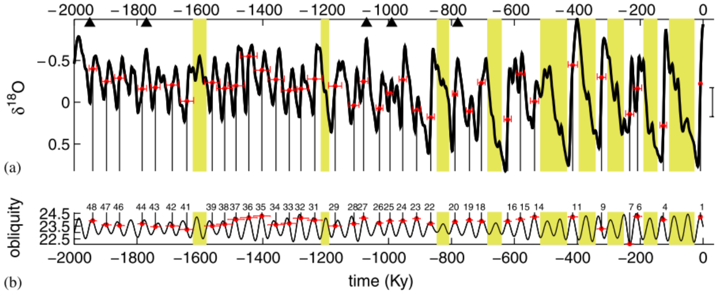
$\delta^{18}O$ content of the last 2Ma



Huybers' Discrete Model

$$\begin{aligned} V_t &= V_{t-1} + \eta_t \qquad \text{and if } V_t \geq T_t \text{ terminate} \\ T_t &= at + b - c\theta'_t \end{aligned}$$

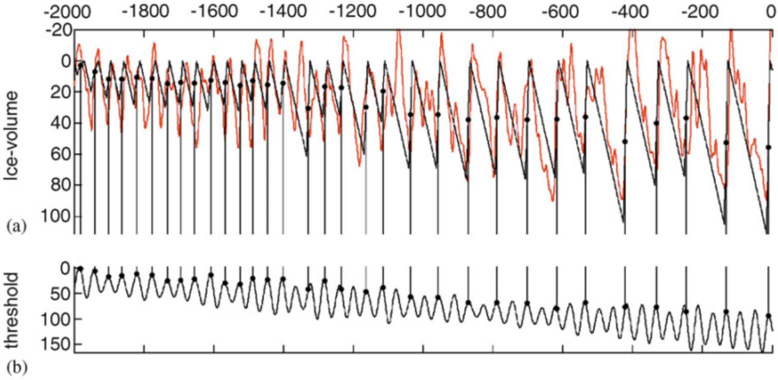
Upon termination, linearly reset V to 0 over 10 Ka

V : ice volume

T : deglaciation threshold

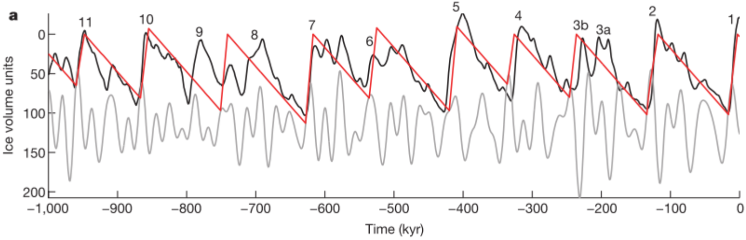
θ' : scaled obliquity

η : ice volume growth rate



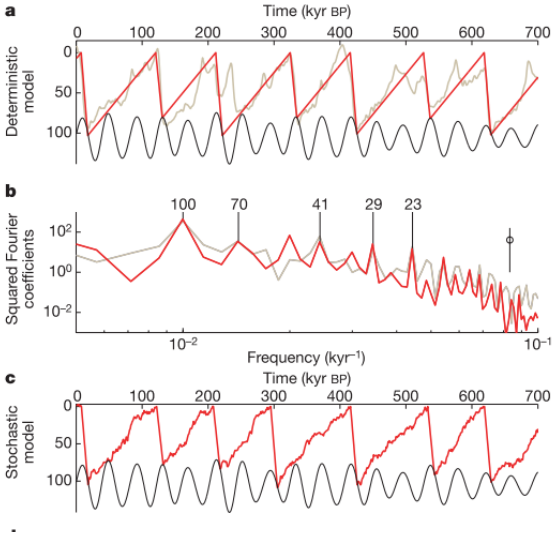
A deterministic run of the model

Huybers, P. Glacial variability over the last two million years: an extended depth-derived agemodel, continuous obliquity pacing, and the Pleistocene progression. *Quaternary Science Reviews*. 2007.



Discrete model with combined forcing

Huybers, P. Combined obliquity and precession pacing of late Pleistocene deglaciations. *Nature*. 2011.



Huybers, P. and Wunsch, C. Obliquity pacing of the late Pleistocene glacial terminations. *Nature*. 2005.

Idealized Model

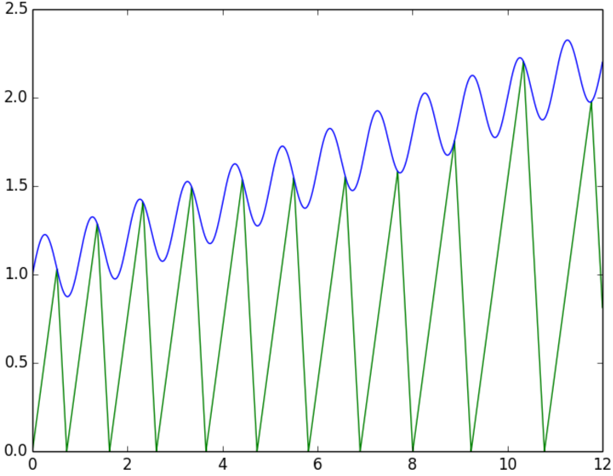
Discrete model:

$$\begin{aligned} V_{t_i} &= V_{t_{i-1}} + \eta_{t_i}\Delta_t \qquad \text{and if } V_{t_i} \geq T_{t_i} \text{ terminate} \\ T_{t_i} &= at_i + b + c\sin(2\pi t_i) \\ \Delta_t &= t_i - t_{i-1} \end{aligned}$$

Continuous model: let $\Delta_t \to 0$.

Let $V_{t_0}(t)$ be the volume with initial condition $V_{t_0}(t_0) = 0$.

Numerical Simulations



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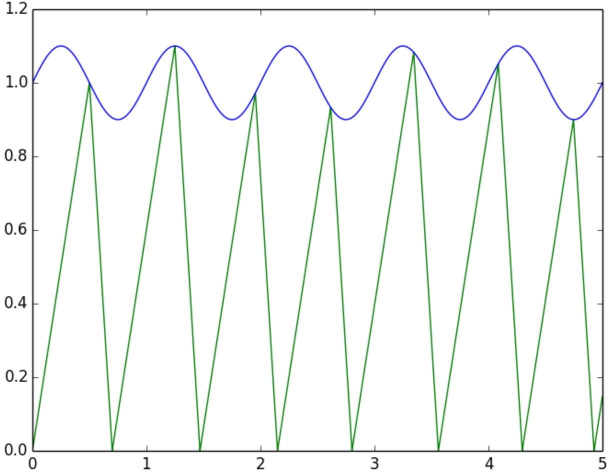


Another model: Neuron Potentials

$$\begin{aligned} \frac{dv}{dt} &= S_0 \\ v(t^+) &= 0 \text{ if } v(t) = T_t \\ T_t &= \theta_0 + \lambda \sin(\omega t + \phi) \end{aligned}$$

v : electric potential

T : firing threshold

J. P. Keener, F. C. Hoppensteadt, and J. Rinzel. Integrate-and-fire models of nerve membrane response to oscillatory input. SIAM Journal on Applied Mathematics, 41:503, 1981.

Reduction to a Periodic Map

Suppose the threshold T is periodic: $T(x+1) = T(x)$.

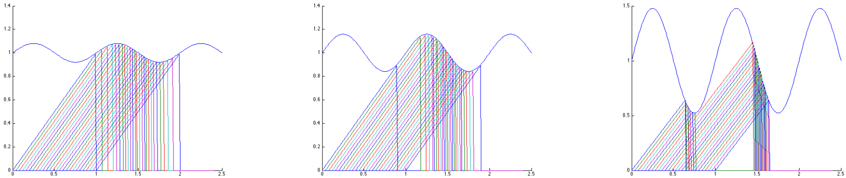
Let $g : \mathbb{R} \to \mathbb{R}$ be the section map sending a termination time t to the next termination time.

$$g(t) = min\{t' > t : V_t(t') = 0\}$$

Then g is also periodic: $g(t+1) = g(t)$.

Reduction to a Periodic Map

The map g can be smooth, continuous, or discontinuous.



Circle Maps

A function $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a circle map.

Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be defined as

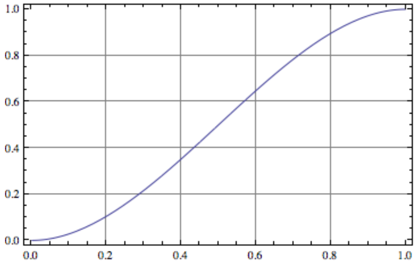
$$\pi(x) = e^{2\pi i x}$$

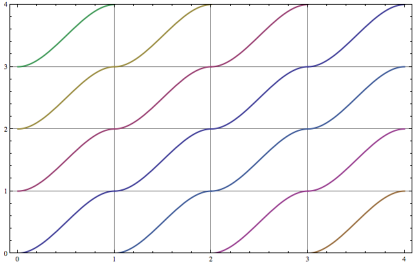
A lift of a circle map is a map $F : \mathbb{R} \to \mathbb{R}$ such that

$$\pi \circ F = f \circ \pi$$

Circle Maps

- There are infinitely many lifts of any circle map f.
- If f is continuous, any two continuous lifts differ by an integer.
- We say a continuous circle map f is orientation preserving if a lift F has the property $F(x) \leq F(y)$ if $x < y$.





Rotation Number

Choose a basepoint $x \in \mathbb{S}^1$ and $x' \in \mathbb{R}$ with $\pi(x') = x$.
Then for f with lift F define

$$\rho(x, f) = \rho(x', F) = \lim_{n \to \infty} \frac{F^n(x') - x'}{n}$$

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"Average" amount of rotation from one iteration of f

Rotation Number

Define the rotation set

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- If f is a diffeomorphism and orientation-preserving, $\rho(f)$ exists uniquely. (Poincaré)

Rotation Number

Define the rotation set

$$\rho(f) = \{\rho(x, f) : x \in \mathbb{S}^1\}$$

- If f is a diffeomorphism and orientation-preserving, $\rho(f)$ exists uniquely. (Poincaré)
- If f is degree one and continuous, $\rho(f)$ is an interval $[\rho_1(f), \rho_2(f)]$. (Ito, 1981)

Average Displacement Set

$$K_n(F) = \frac{F^n - id}{n}(\mathbb{R}) = \frac{F^n - id}{n}([0,1])$$

$$K(F) = \bigcap_{n \in \mathbb{N}} K_n(F)$$

Rotation Number

- For a degree one, continuous circle map f with lift F,

  $p/q \in \rho(f) \quad \Leftrightarrow \quad$ There exists point $x \in \mathbb{R}$ with $F^q(x) = x + p$

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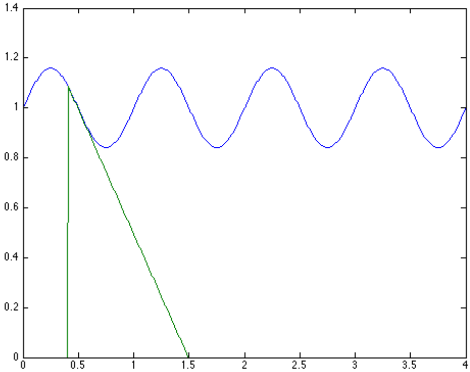
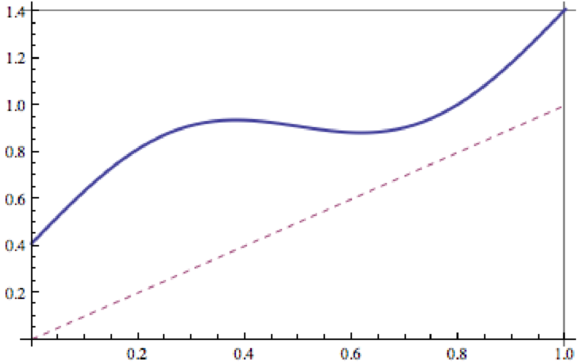
- $K(F) = \rho(F)$

Standard family of circle maps

$$f(x) = x + b + \frac{\omega}{2\pi}\sin(2\pi x) \mod 1$$



Standard family of circle maps



Discontinuous Rotations

What holds true for discontinuous rotations?

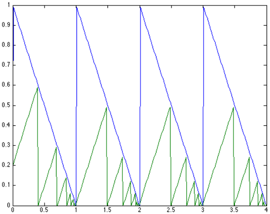
Discontinuous Rotations

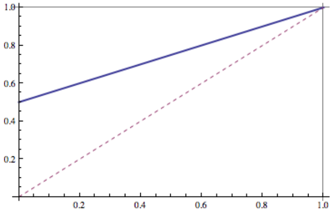
What holds true for discontinuous rotations?

- Existence and uniqueness if f is orientation preserving. (Brette, 2003; Kozaykin, 2005)
- If there exists point z with $f^q(z) = z$, $p/q \in \rho(f)$

Discontinuous Rotations

- $p/q \in \rho(f)$ does not imply the existence of a periodic point: $f(x) = (1/2)x + 1/2$





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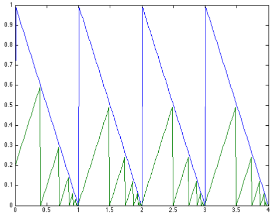

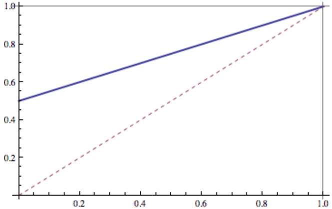

- BUT, if $p/q \in \rho(f)$, orbits will tend towards a (possibly missing) periodic orbit.

Relations on $\mathbb{S}^1$

A relation on $\mathbb{S}^1$ is a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.
The analogue of an iteration is an *orbit* of a relation f:

$$\{...x_{-1}, x_0, x_1, x_2, ...\} \text{ such that } (x_i, x_{i+1}) \in f.$$

Rotation set is:

$$\rho(f) = \rho(F) = \left\{ \lim_{n\to\infty} \frac{x_n - x_0}{n}, (x_0, x_1, x_2, ...) \text{ is an orbit of F } \right\}$$

Closed, Connected Relations

What holds true for rotation numbers of closed, connected relations?

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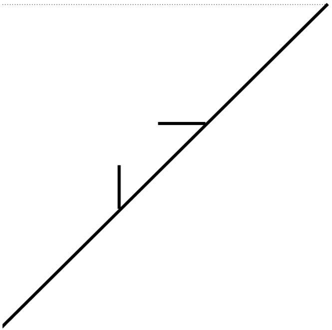
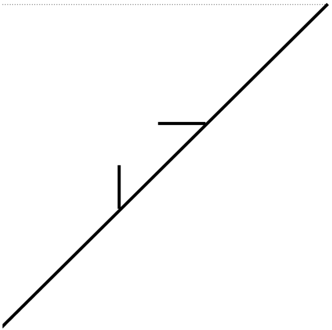
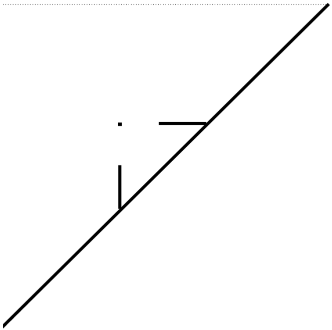


Closed, Connected Relations

What holds true for rotation numbers of closed, connected relations?

- Connected relations might not stay connected upon iteration!



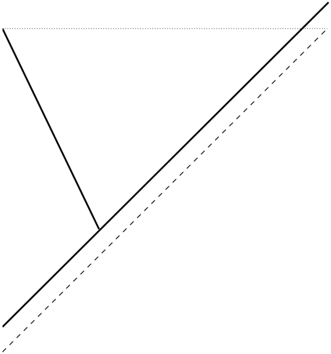


Closed, Connected Relations

- The rotation set is not always a closed interval.

Closed, Connected Relations

- The rotation set is not always a closed interval.
- Consider a relation consisting of two lines:
  $x + \alpha$, and $1 - \alpha x$.



Closed, Connected Relations

There is one orbit starting at 0 that moves up by 1 every time, with rotation number 1.

All other orbits move at most $1 + \alpha$ after 2 moves, with rotation number in $[\alpha, (1 + \alpha)/2]$.

Closed, Connected Relations

Can these two types of orbits mix?

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$m_1\alpha$

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$m_1\alpha$
$n_1 - \alpha(m_1\alpha)$

Closed, Connected Relations

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$m_1\alpha$
$n_1 - \alpha(m_1\alpha)$
$n_1 - \alpha(m_1\alpha) + m_2\alpha$

Closed, Connected Relations

Can these two types of orbits mix?

$m_1\alpha$
$n_1 - \alpha(m_1\alpha)$
$n_1 - \alpha(m_1\alpha) + m_2\alpha$
$n_2 - \alpha(n_1 - \alpha(m_1\alpha) + m_2\alpha)$

Closed, Connected Relations

Can these two types of orbits mix?

$m_1\alpha$
$n_1 - \alpha(m_1\alpha)$
$n_1 - \alpha(m_1\alpha) + m_2\alpha$
$n_2 - \alpha(n_1 - \alpha(m_1\alpha) + m_2\alpha)$
$n_2 - \alpha(n_1 - \alpha(m_1\alpha) + m_2\alpha)... = N$?

Closed, Connected Relations

Can these two types of orbits mix?

$m_1\alpha$
$n_1 - \alpha(m_1\alpha)$
$n_1 - \alpha(m_1\alpha) + m_2\alpha$
$n_2 - \alpha(n_1 - \alpha(m_1\alpha) + m_2\alpha)$
$n_2 - \alpha(n_1 - \alpha(m_1\alpha) + m_2\alpha)... = N?$

This is a polynomial in α with integer coefficients. If α is transcendental, the equation can not be satisfied.

What do we know?

Orientation-preserving $\Rightarrow$ unique rotation number

Rational rotation number $\Leftrightarrow$ periodic point

Conjectures

Conjecture: If connectedness is preserved, the rotation set is a closed interval, and $\rho(F) = K(F)$.

- (need to modify Ito's proof that rotation sets are closed)

Conjecture: $\rho(F) = K(F)$

Conjecture: rotation set for backwards (inverse) iterations will be the same.